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Zagreb-Based Indices of Prime Coprime Graph for Integers Modulo Power of Primes

Abdurahim [1], Romdhini, M. U.* [1], Qudsi, J. [2], and Husain, S. K. S. [3,4]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram 83125, Indonesia*
[2] *Department of Statistics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram 83125, Indonesia*
[3] *Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[4] *Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*

E-mail: mamika@unram.ac.id
**Corresponding author*

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Abstract

The prime coprime graph is a graph in which every pair of vertices is adjacent if and only if the greatest common divisor of the order of both vertices is equal to one or relatively prime. This study aims to analyze the general formula of the first and second Zagreb, and hyper Zagreb indices of prime coprime graphs for a group of integers modulo power of primes. The correlation between these indices is found to have a strong significant correlation; in other words, if one index increases, then the other indices also increase with a coefficient correlation greater than 0.95. We also demonstrate this correlation in one equation and our results improve the understanding of Zagreb-based indices.

Keywords: first Zagreb index; second Zagreb index; hyper Zagreb index; prime coprime graph; integers modulo.

1 Introduction

Topological indices are referred to as molecular structure descriptors and can be used to describe the chemical properties of molecules. The chemical structure of a molecule can be quantitatively related to its physical characteristics, chemical reactivity, or biological action using topological indices. Harold Wiener, a chemist, used a topological index for the first time in 1947 to determine the physical characteristics of paraffins, which were alkanes with various topological descriptors [23]. This topic has various applications, as shown in several previous literature. Gowtham et al. [9] studied the physicochemical characteristics of benzenoid hydrocarbons and described their structure with a symmetric division deg index of a graph. The first Zagreb index is correlated with different quantities of interest in Chemical Graph Theory, as demonstrated by Gutman and Das [11]. Wang and Belardo [24] presented the application of the first Zagreb index. In addition, the determination of the entropy of a molecular structure of diamonds with degree-based topological indices has been discussed by [15].

Gutman and Trinajstic [13] first introduced the first and second Zagreb indices in 1972. Later in 1975, the second Zagreb index was also presented in [12] and is a topological index based on vertex degrees. Moreover, the Zagreb-based indices have extended the discussion to other indices, including the Hyper-Zagreb index. It was created in 2013 by Shirdel et al. [22] and is just the general sum-connectivity index introduced in [26]. Reverse topological indices are also an interesting topic; readers may refer to [8] for a discussion of the reverse Zagreb index of a graph.

In advanced graph theory, research has been significantly developed while considering the combination of graph and algebraic structures, known as graph representation in algebra. This topic has attracted the attention of researchers in recent years, as it has been proven by various conducted studies. Research on topological indices is also associated with the graph defined in the group or ring such as the non-zero divisor graph [25], prime ideal graph [21], the commuting and non-commuting graphs [20], and the prime graph [16]. Bello et al. [4] studied the Wiener, first Zagreb, and second Zagreb indices of the order product prime graph. Several properties of the coprime graph of a group can be found in [17], as well as Abdurahim et al. [1] presented the coprime graph for integer groups.

In addition to those graphs, Adhikari and Banerjee [3] introduced the concept of the prime coprime graph of a finite group G. It is denoted by Γ_G, with the vertex set as all elements of the group G, where two distinct vertices u and v are adjacent whenever the greatest common divisor of order of both vertices is 1 or a prime number, $(|u|, |v|) = 1$ or $(|u|, |v|) = p$, where p is a prime number [3]. Considering this invention, a research problem is encountered when investigating the topological index of Γ_G. The definition of Γ_G triggers us to focus on the integers modulo power prime as investigated in [14]. However, a limited exploration of the Zagreb-based indices of this graph bridges the gap addressed in this study. Therefore, we devote this research to prime coprime graphs for groups $\mathbb{Z}_n$ and is denoted by $\Gamma_{\mathbb{Z}_n}$, focusing on Zagreb-based indices including the first, second and hyper-Zagreb indices with $n = p^k$ and integer $k \geq 2$.

This paper is organized as follows. The basic definition and theory are presented in Section 2. In the next section, we describe the new findings as the main result of this research, including the statistical views of the topological indices formulas obtained. The summary of this research is shown in Section 4.

2 Preliminaries

The basic notation and definition are defined in this section. We recall the notation of the prime coprime graphs for the $\mathbb{Z}_n$ groups as $\Gamma_{\mathbb{Z}_n}$. The set of vertices of $\Gamma_{\mathbb{Z}_n}$ is denoted by $V(\Gamma_{\mathbb{Z}_n})$ and the set of edges is $E(\Gamma_{\mathbb{Z}_n})$. Two distinct vertices u and v are adjacent if and only if an edge $uv \in E(\Gamma_{\mathbb{Z}_n})$.

Now, divide $\mathbb{Z}_n$ for $n = p^k$ with the prime p and the integer $k \geq 2$ into two sets as follows:

$$V_1 = \{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\}, \text{ and}$$
$$V_2 = \{0, 1, 2, 3, \ldots, p^k - 1\} \backslash V_1,$$

where $V_1 \cup V_2 = \mathbb{Z}_n$. Let $\deg(v)$ be the degree of the vertex v. The following result is essential to calculate the Zagreb-based indices in the next section.

Theorem 2.1. [2] *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of the $\mathbb{Z}_n$ groups. If $n = p^k$ with p prime and integer $k \geq 2$, then the degree of vertex v in $\Gamma_{\mathbb{Z}_n}$ is,*

$$\deg(v) = \begin{cases} p^k - 1, & v \in V_1, \\ p, & v \in V_2. \end{cases}$$

Furthermore, here we write the definition of the Zagreb-based indices. The first Zagreb index of $\Gamma_{\mathbb{Z}_n}$ is a square of the summation of all vertex degrees in $\Gamma_{\mathbb{Z}_n}$ [11] and formulated by,

$$M_1(\Gamma_{\mathbb{Z}_n}) = \sum_{v \in V(\Gamma_{\mathbb{Z}_n})} (\deg(v))^2. \tag{1}$$

The second Zagreb index of $\Gamma_{\mathbb{Z}_n}$ is the product of the degree of two distinct adjacent vertices [6] and is denoted by,

$$M_2(\Gamma_{\mathbb{Z}_n}) = \sum_{uv \in E(\Gamma_{\mathbb{Z}_n})} \deg(u) \cdot \deg(v). \tag{2}$$

The next type of Zagreb-based indices is hyper-Zagreb, which is the square of the summation of the degree of two distinct adjacent vertices [22],

$$HM(\Gamma_{\mathbb{Z}_n}) = \sum_{uv \in E(\Gamma_{\mathbb{Z}_n})} (\deg(u) + \deg(v))^2. \tag{3}$$

Moreover, for computation purposes, we need to define the following notation. This article uses the notation S_p, where S_p represents the arithmetic sum of the first p terms.

This research deals with two mathematical fields: graph theory and group theory. We conducted the research objective, which is to formulate the Zagreb-based indices of the prime coprime graph for group $\mathbb{Z}_n$. Then, brief descriptions of the methods used to achieve this goal are explained. We study the degree of every vertex in $\Gamma_{\mathbb{Z}_n}$ as stated in Theorem 2.1. Then, we can conclude the connectivity of the graph structure. Using this property, we can construct the calculation of Zagreb-based indices that include the first, second, and hyper Zagreb indices using (1), (2) and (3). In addition to analyzing the index, it is also possible to investigate the relationship between those indices.

Based on Theorem 2.1, $\Gamma_{\mathbb{Z}_n}$, for $n = p^k$ with p prime and integer $k \geq 2$, can be classified as a bidegreed graph with vertices having exactly two degrees. In chemistry, alkanes are a bidegreed graph. Alkanes are the most basic and simple hydrocarbons present in organic compounds; they consist solely of hydrogen (H) and carbon (C) atoms.

Let a and b be the minimum and maximum degrees of vertices in a bidegreed graph Γ, respectively. Therefore, we have the lower bound of Γ based on Das [5] and the improvement of the sharper bound from [18] as follows:

Theorem 2.2. [18] *Let Γ be a bidegreed graph with n vertices and m edges. Then,*

$$M_1(\Gamma) \geq b^2 + \frac{(2e-b)^2}{n-1}. \tag{4}$$

Moreover, the relation between the first, second and hyper-Zagreb indices of Γ as a bidegreed graph can be found in [19] and [7] as stated below.

Theorem 2.3. [19] *Let Γ be a bidegreed graph. Then,*

$$M_2(\Gamma) \leq \frac{1}{2}\sqrt{M_1(\Gamma)}.$$

Theorem 2.4. [7] *Let Γ be a bidegreed graph with n vertices and m edges. Then,*

$$HM(\Gamma) \leq 2M_2(\Gamma) + (a+b)M_1(\Gamma) - 2eab.$$

3 Results

This section investigates the computations of the main results, which are the Zagreb-based indices of the prime coprime graph associated with the first, second, and hyper-Zagreb indices. We know that $\Gamma_{\mathbb{Z}_n}$ is a simple graph without loops or multiple edges. It is also a connected graph. Therefore, we can apply those Zagreb-based indices to $\Gamma_{\mathbb{Z}_n}$.

We begin with the result on the total edges of $\Gamma_{\mathbb{Z}_n}$ as given below, which is beneficial for comparing with similar studies or indices in the literature. The sum of the degrees of all the vertices in any graph is equal to twice the number of its edges [10]. In other words, the number of edges in any graph is half the sum of the degrees of the vertex.

Theorem 3.1. *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the number of edges of $\Gamma_{\mathbb{Z}_n}$ is,*

$$e = |E(\Gamma_{\mathbb{Z}_n})| = \frac{1}{2}\left(2p^{k+1} - p^2 - p\right).$$

Proof. Based on the vertices partition in the proof of Theorem 2.1, it is obtained that $|V_1| = p$ and $|V_2| = p^k - p$. Furthermore, it is known that $\deg(v_1) = p^k - 1$ and $\deg(v_2) = p$ for any $v_1 \in V_1$ and $v_2 \in V_2$. Observe that,

$$
\begin{aligned}
e &= |E(\Gamma_{\mathbb{Z}_n})| \\
&= \frac{1}{2}\sum_{v\in V}\deg(v) \\
&= \frac{1}{2}\left(\sum_{v_1\in V_1}\deg(v_1) + \sum_{v_2\in V_2}\deg(v_2)\right) \\
&= \frac{1}{2}\left(p\cdot(p^k-1) + (p^k-p)\cdot p\right) \\
&= \frac{1}{2}\left(2p^{k+1}-p^2-p\right).
\end{aligned}
$$

Thus, the number of edges in the prime coprime graph $\Gamma_{\mathbb{Z}_n}$ with $n = p^k$ given by $\frac{1}{2}\left(2p^{k+1}-p^2-p\right)$. □

We first discuss the first Zagreb index.

Theorem 3.2. *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the first Zagreb index of $\Gamma_{\mathbb{Z}_n}$ is,*

$$M_1(\Gamma_{\mathbb{Z}_n}) = p\left(p^{2k}+p^{k+1}-2p^k-p^2+1\right).$$

Proof. According to Theorem 2.1, the set of vertices of $\Gamma_{\mathbb{Z}_n}$ can be partitioned into two sets, V_1 and V_2. It should be noted that the number of vertices in V_1 is equal to p. Based on Theorem 2.1, the degree of each vertex in V_1 is p^{k-1}. Consequently, we immediately have,

$$\sum_{v\in V_1(\Gamma_{\mathbb{Z}_n})}(\deg(v))^2 = p(p^k-1)^2.$$

Meanwhile, the cardinality of V_2 is $p^k - p$ and conforming from Theorem 2.1, the degree of every vertex in V_2 is p. It follows that,

$$\sum_{v\in V_2(\Gamma_{\mathbb{Z}_n})}(\deg(v))^2 = (p^k-p)p^2.$$

Therefore, we can compute the first Zagreb index following (1) as given below,

$$
\begin{aligned}
M_1(\Gamma_{\mathbb{Z}_n}) &= \sum_{v\in V_1(\Gamma_{\mathbb{Z}_n})}(\deg(v))^2 + \sum_{v\in V_2(\Gamma_{\mathbb{Z}_n})}(\deg(v))^2 \\
&= p(p^k-1)^2 + (p^k-p)p^2 \\
&= p\left(p^{2k}+p^{k+1}-2p^k-p^2+1\right).
\end{aligned}
$$

□

The next theorem is the computation of the second Zagreb index of $\Gamma_{\mathbb{Z}_n}$.

Theorem 3.3. *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the second Zagreb of $\Gamma_{\mathbb{Z}_n}$ is,*

$$M_2(\Gamma_{\mathbb{Z}_n}) = \frac{1}{2}p(p^k-1)\left(3p^{k+1}-p^k-2p^2-p+1\right).$$

Proof. By the same argument of Theorem 3.2, then the vertex set of $\Gamma_{\mathbb{Z}_n}$ can be partitioned into two sets V_1 and V_2. We consider three cases in this proof. The first is considering edges between two distinct vertices in V_1. The second case focuses on edges between vertices in V_1 to V_2 and the third case observes the edges between vertices in V_2 to V_1. The second and third cases are distinguished to simplify the construction of the proof by utilizing arithmetic series.

Case 1: We first analyze edges connecting two distinct vertices in V_1. Let u_1, v_1 be the two vertices in V_1 with $u_1 \neq v_1$. According to Theorem 2.1, we derive $\deg(u_1) = \deg(v_1) = p^k - 1$. As a consequence of Theorem 3.2, $\Gamma_{\mathbb{Z}_n}$ for V_1 is a complete graph. It follows that the number of edges is S_{p-1} which implies,

$$\sum_{u_1v_1 \in E(\Gamma_{\mathbb{Z}_n})} \deg(u_1)\deg(v_1) = S_{p-1} \cdot \left(p^k - 1\right)\left(p^k - 1\right)$$
$$= \frac{1}{2}(p-1)\,p \cdot \left(p^k - 1\right)\left(p^k - 1\right).$$

Case 2: We now observe edges connecting vertices in V_1 to V_2. The second case when $u_1 \in V_1$ and $v_2 \in V_2$, following Theorem 2.1, we then derive $\deg(u_1) = p^k - 1$ and $\deg(v_2) = p$. The number of edges connecting vertices in V_1 to V_2 is equal to the first p terms of an arithmetic series, S_p, multiplied by the number of vertices V_2 located between any two vertices in V_1, $p^{k-1} - 1$. We have a number $\left(p^{k-1} - 1\right) S_p$ edges connecting vertices in V_1 and vertices in V_2. Thus, we can state that,

$$\sum_{u_1v_2 \in E(\Gamma_{\mathbb{Z}_n})} \deg(u_1)\deg(v_2) = (p^{k-1} - 1)S_p \cdot \left(p^k - 1\right)p$$
$$= \frac{1}{2}\left(p^{k-1} - 1\right)\left(p^2 + p\right)\left(p^k - 1\right).$$

Case 3: This case considers edges connecting vertices in V_2 to V_1. Let $u_2 \in V_2$ and $v_1 \in V_1$. By Theorem 3.2 we obtain $\deg(u_2) = p$ and $\deg(v_1) = p^k - 1$. We notice that there are $\left(p^{k-1} - 1\right) S_{p-1}$ edges connecting V_2 and V_1. Using these facts, we can write,

$$\sum_{u_2v_1 \in E(\Gamma_{\mathbb{Z}_n})} \deg(u_1)\deg(v_2) = (p^{k-1} - 1)S_{p-1} \cdot \left(p^k - 1\right)p$$
$$= \frac{1}{2}\left(p^{k-1} - 1\right)\left(p^2 - p\right)\left(p^k - 1\right).$$

Based on those three cases and (2), it can be concluded that,

$$M_2\left(\Gamma_{\mathbb{Z}_n}\right) = \frac{1}{2}p(p^k - 1)\left(3p^{k+1} - p^k - 2p^2 - p + 1\right).$$

□

In the following theorem, we obtain the hyper-Zagreb index of $\Gamma_{\mathbb{Z}_n}$.

Theorem 3.4. *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of the $\mathbb{Z}_n$ groups. If $n = p^k$ with p prime and integer $k \geq 2$, then the hyper-Zagreb index of $\Gamma_{\mathbb{Z}_n}$ is,*

$$HM\left(\Gamma_{\mathbb{Z}_n}\right) = p^{3k+1} + 3p^{2k+2} - 4p^{2k+1} - p^{k+3} - 4p^{k+2} + 5p^{k+1} - p^4 + 2p^3 + p^2 - 2p.$$

Proof. By the same argument of proofing part of Theorem 3.2, then,

Case 1: For two distinct vertices in V_1, u_1 and v_1 with $u_1 \neq v_1$, we have,

$$\begin{aligned}\sum_{u_1v_1 \in E(\Gamma_{\mathbb{Z}_n})} (\deg(u_1) + \deg(v_1))^2 &= S_{p-1} \cdot \left(\left(p^k - 1\right) + \left(p^k - 1\right)\right)^2 \\ &= \frac{1}{2}(p-1)\,p \cdot \left(2\left(p^k - 1\right)\right) \\ &= 2p^{2k+2} - 4p^{k+2} + 2p^2 - 2p^{2k+1} + 4p^{k+1} - 2p.\end{aligned}$$

Case 2: For edges connecting vertices from V_1 to V_2 with $u_1 \in V_1$ and $v_2 \in V_2$, we obtain,

$$\begin{aligned}\sum_{u_1v_2 \in E(\Gamma_{\mathbb{Z}_n})} (\deg(u_1) + \deg(v_2))^2 &= \left(p^{k-1} - 1\right) S_p \cdot \left(\left(p^k - 1\right) + p\right)^2 \\ &= \frac{1}{2}\left(p^{k-1} - 1\right)\left(p^2 + p\right)\left(p^k + p - 1\right)^2 \\ &= \frac{1}{2}p^{k+1} - \frac{1}{2}p^{k+2} - \frac{1}{2}p^{k+3} - \frac{1}{2}p^{2k+1} + \frac{1}{2}p^{2k+2} + \frac{1}{2}p^{3k+1} \\ &\quad + \frac{1}{2}p^{3k} - p^{2k} + \frac{1}{2}p^k - \frac{1}{2}p^4 + \frac{1}{2}p^3 + \frac{1}{2}p^2 - \frac{1}{2}p.\end{aligned}$$

Case 3: For edges connecting V_2 to V_1 where $u_2\, inV_2$ and $v_1 \in V_1$, we get,

$$\begin{aligned}\sum_{u_2v_1 \in E(\Gamma_{\mathbb{Z}_n})} (\deg(u_2) + \deg(v_1))^2 &= \left(p^{k-1} - 1\right) S_{p-1} \cdot \left(p + \left(p^k - 1\right)\right)^2 \\ &= \frac{1}{2}\left(p^{k-1} - 1\right)\left(p^2 - p\right)\left(p^k + p - 1\right)^2 \\ &= \frac{1}{2}p^{k+1} + \frac{1}{2}p^{k+2} - \frac{1}{2}p^{k+3} - \frac{3}{2}p^{2k+1} + \frac{1}{2}p^{2k+2} + \frac{1}{2}p^{3k+1} \\ &\quad - \frac{1}{2}p^{3k} + p^{2k} - \frac{1}{2}p^k - \frac{1}{2}p^4 + \frac{3}{2}p^3 - \frac{3}{2}p^2 + \frac{1}{2}p.\end{aligned}$$

By summation of Case 1, Case 2, Case 3 and by (3), we conclude that,

$$HM(\Gamma_{\mathbb{Z}_n}) = p^{3k+1} + 3p^{2k+2} - 4p^{2k+1} - p^{k+3} - 4p^{k+2} + 5p^{k+1} - p^4 + 2p^3 + p^2 - 2p.$$

□

An example of a prime coprime graph that illustrates the main findings of this study is presented below. The calculation of Zagreb-based indices is involved. We take $p = 3$ and $k = 2$, the graph construction of $\Gamma_{\mathbb{Z}_9}$ is shown in Figure 1.

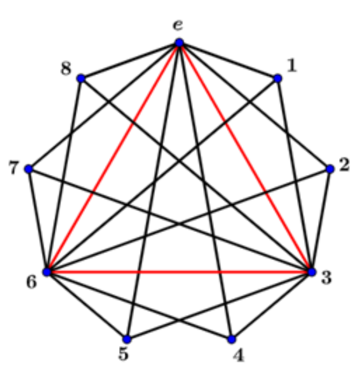

Figure 1: Pime Coprime Graph for $\mathbb{Z}_9$.

Based on the information of the graph, we have the degree of every vertex in $\mathbb{Z}_9$ is $deg(e) = deg(3) = deg(6) = 8$, $deg(1) = deg(2) = deg(4) = deg(5) = deg(7) = deg(8) = 3$. Then, we can calculate the first Zagreb index of $\mathbb{Z}_9$ as given below,

$$\begin{aligned} M_1\left(\Gamma_{\mathbb{Z}_9}\right) &= \sum_{v \in V\left(\Gamma_{\mathbb{Z}_9}\right)} (\deg(v))^2 \\ &= (\deg(0))^2 + (\deg(1))^2 + (\deg(2))^2 + (\deg(3))^2 + (\deg(4))^2 + (\deg(5))^2 \\ &\quad + (\deg(6))^2 + (\deg(7))^2 + (\deg(8))^2 \\ &= 8^2 + 3^2 + 3^2 + 8^2 + 3^2 + 3^2 + 8^2 + 3^2 + 3^2 \\ &= 246. \end{aligned}$$

Notice that the milestone of the second and hyper-Zagreb definitions is the adjacency between two vertices. Based on the edges in $\Gamma_{\mathbb{Z}_9}$, then,

$$\begin{aligned} M_2\left(\Gamma_{\mathbb{Z}_9}\right) &= \sum_{uv \in E\left(\Gamma_{\mathbb{Z}_9}\right)} \deg(u) \cdot \deg(v) \\ &= \deg(0)\cdot\deg(1) + \deg(0)\cdot\deg(2) + \deg(0)\cdot\deg(3) + \deg(0)\cdot\deg(4) + \deg(0)\cdot\deg(5) \\ &\quad + \deg(0)\cdot\deg(6) + \deg(0)\cdot\deg(7) + \deg(0)\cdot\deg(8) + \deg(1)\cdot\deg(3) + \deg(1)\cdot\deg(6) \\ &\quad + \deg(2)\cdot\deg(3) + \deg(2)\cdot\deg(6) + \deg(3)\cdot\deg(4) + \deg(3)\cdot\deg(5) + \deg(3)\cdot\deg(6) \\ &\quad + \deg(3)\cdot\deg(7) + \deg(3)\cdot\deg(8) + \deg(4)\cdot\deg(6) + \deg(5)\cdot\deg(6) + \deg(6)\cdot\deg(7) \\ &\quad + \deg(6)\cdot\deg(8) \\ &= 8\cdot 3 + 8\cdot 3 + 8\cdot 8 + 8\cdot 3 + 8\cdot 3 + 8\cdot 8 + 8\cdot 3 + 8\cdot 3 + 3\cdot 8 + 3\cdot 8 + 3\cdot 8 + 3\cdot 8 \\ &\quad + 8\cdot 3 + 8\cdot 3 + 8\cdot 8 + 8\cdot 3 + 8\cdot 3 + 3\cdot 8 + 3\cdot 8 + 8\cdot 3 + 8\cdot 3 \\ &= 624, \end{aligned}$$

and

$$\begin{aligned} HM\left(\Gamma_{\mathbb{Z}_9}\right) &= \sum_{uv \in E\left(\Gamma_{\mathbb{Z}_9}\right)} (\deg(u) + \deg(v))^2 \\ &= (\deg(0) + \deg(1))^2 + (\deg(0) + \deg(2))^2 + (\deg(0) + \deg(3))^2 + (\deg(0) + \deg(4))^2 \\ &\quad + (\deg(0) + \deg(5))^2 + (\deg(0) + \deg(6))^2 + (\deg(0) + \deg(7))^2 + (\deg(0) + \deg(8))^2 \\ &\quad + (\deg(1) + \deg(3))^2 + (\deg(1) + \deg(6))^2 + (\deg(2) + \deg(3))^2 + (\deg(2) + \deg(6))^2 \\ &\quad + (\deg(3) + \deg(4))^2 + (\deg(3) + \deg(5))^2 + (\deg(3) + \deg(6))^2 + (\deg(3) + \deg(7))^2 \\ &\quad + (\deg(3) + \deg(8))^2 + (\deg(4) + \deg(6))^2 + (\deg(5) + \deg(6))^2 + (\deg(6) + \deg(7))^2 \\ &\quad + (\deg(6) + \deg(8))^2 \\ &= (8+3)^2 + (8+3)^2 + (8+8)^2 + (8+3)^2 + (8+3)^2 + (8+8)^2 + (8+3)^2 + (8+3)^2 \\ &\quad + (3+8)^2 + (3+8)^2 + (3+8)^2 + (3+8)^2 + (8+3)^2 + (8+3)^2 + (8+8)^2 + (8+3)^2 \\ &\quad + (8+3)^2 + (3+8)^2 + (3+8)^2 + (8+3)^2 + (8+3)^2 \\ &= 2946. \end{aligned}$$

In the next stage, let us proceed under Theorems 3.2, 3.3, and 3.4 for the first, second, and hyper-Zagreb indices by substituting the value of $p = 3$ and $k = 2$ into the formula of those theorems. Hence, we get the respective index values as follows:

$$\begin{aligned}
M_1(\Gamma_{\mathbb{Z}_9}) &= p\left(p^{2k}+p^{k+1}-2p^k-p^2+1\right)\\
&= 3\left(3^{2\cdot 2}+3^{2+1}-2\cdot 3^2-3^2+1\right)\\
&= 3\left(81+27-18-9+1\right)\\
&= 246,\\
M_2(\Gamma_{\mathbb{Z}_9}) &= \frac{1}{2}p(p^k-1)\left(3p^{k+1}-p^k-2p^2-p+1\right)\\
&= \frac{1}{2}\cdot 3(3^2-1)\left(3\cdot 3^{2+1}-3^2-2\cdot 3^2-3+1\right)\\
&= \frac{1}{2}\cdot 3\cdot 8\cdot(81-9-18-3+1)\\
&= 624,\\
HM(\Gamma_{\mathbb{Z}_9}) &= p^{3k+1}+3p^{2k+2}-4p^{2k+1}-p^{k+3}-4p^{k+2}+5p^{k+1}-p^4+2p^3+p^2-2p\\
&= 3^{3\cdot 2+1}+3\cdot 3^{2\cdot 2+2}-4\cdot 3^{2\cdot 2+1}-3^{2+3}-4\cdot 3^{2+2}+5\cdot 3^{2+1}-3^4+2\cdot 3^3+3^2-2\cdot 3\\
&= 2.187+2.187-972-243-324+135-81+54+9-6\\
&= 2946,
\end{aligned}$$

and conforming to the results.

Furthermore, $\Gamma_{\mathbb{Z}_n}$ is a bidegreed graph since it has exactly two options of vertex degree, p and p^k-1. Hence, according to the previous literature in Theorem 2.2, our finding result in Theorem 3.2 complies with it, and the statement is given below.

Theorem 3.5. *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n=p^k$ with p prime and integer $k\geq 2$, then,*

$$M_1(\Gamma_{\mathbb{Z}_n})\geq b^2+\frac{(2e-b)^2}{n-1}.$$

Proof. There are $n=p^k$ vertices and by Theorem 3.1, we have $e=\frac{1}{2}\left(2p^{k+1}-p^2-p\right)$. Meanwhile, from Theorem 2.1 we know that $a=p$ and $b=p^k-1$. By Theorem 3.2, we have,

$$\begin{aligned}
M_1(\Gamma_{\mathbb{Z}_n}) &= p\left(p^{2k}+p^{k+1}-2p^k-p^2+1\right)\\
&= p((p^k-1)^2+p^{k+1}-p^2)\\
&= (p^k-1)^2+(p-1)(p^k-1)^2+p^{k+2}-p^3\\
&= (p^k-1)^2+\frac{(p-1)(p^k-1)^3+(p^{k+2}-p^3)(p^k-1)}{p^k-1}\\
&= (p^k-1)^2+\frac{(p-1)(p^k-1)^3+p^2(p^k-1)^2-(p-1)p^2(p^k-1)}{p^k-1}\\
&\geq (p^k-1)^2+\frac{(2p^{k+1}-p^2-p)^2+(p^k-1)^2-2(2p^{k+1}-p^2-p)(p^k-1)}{p^k-1}\\
&= \left(p^k-1\right)^2+\frac{\left(\left(2p^{k+1}-p^2-p\right)-\left(p^k-1\right)\right)^2}{p^k-1}\\
&= b^2+\frac{(2e-b)^2}{n-1}.
\end{aligned}$$

It is conforming the right side of inequality (4) in Theorem 2.2 and the proof is complete. □

However, Theorem 2.3 does not hold for the results of Theorems 3.2 and 3.3. From the previous example where $p = 3$ and $n = 2$, we get $M_1(\Gamma_{\mathbb{Z}_9}) = 246$ and $M_2(\Gamma_{\mathbb{Z}_9}) = 624$. These values do not meet the necessary condition of Theorem 2.3, since we have,

$$M_2(\Gamma_{\mathbb{Z}_9}) > \frac{1}{2}\sqrt{M_1(\Gamma_{\mathbb{Z}_9})}.$$

In addition, the relationship between the Zagreb-based $\Gamma_{\mathbb{Z}_n}$ indices in this study is clearly stated in the following theorem. We can declare our finding results in Theorem 3.4 to comply with Theorem 2.4 in the following way.

Theorem 3.6. *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of the $\mathbb{Z}_n$ groups. Then,*

$$HM(\Gamma_{\mathbb{Z}_n}) = 2M_2(\Gamma_{\mathbb{Z}_n}) + (a+b)M_1(\Gamma_{\mathbb{Z}_n}) - 2eab.$$

Proof. By the same argument of the proof or Theorem 3.5 with $n = p^k$, $e = \frac{1}{2}\left(2p^{k+1} - p^2 - p\right)$, $a = p$ and $b = p^k - 1$, in accordance with Theorem 3.4, we have,

$$\begin{aligned}
HM(\Gamma_{\mathbb{Z}_n}) &= p^{3k+1} + 3p^{2k+2} - 4p^{2k+1} - p^{k+3} - 4p^{k+2} + 5p^{k+1} - p^4 + 2p^3 + p^2 - 2p \\
&= p(p^k-1)(p^{2k} + 2p^{k+1} - 3p^k - 2p^2 + 2) + p^2(p^{2k} + p^{k+1} - 2p^k - p^2 + 1) \\
&= p(p^k-1)(p^{k+1} - p^k - p^2 + 1) + (p^k + p - 1)p(p^{2k} + p^{k+1} - 2p^k - p^2 + 1) \\
&= p(p^k-1)(3p^{k+1} - p^k - 2p^2 - p + 1) + (p + p^k - 1)p(p^{2k} + p^{k+1} - 2p^k - p^2 + 1) \\
&\quad - (2p^{k+1} - p^2 - p)p(p^k - 1) \\
&= 2M_2(\Gamma_{\mathbb{Z}_n}) + (a+b)M_1(\Gamma_{\mathbb{Z}_n}) - 2eab,
\end{aligned}$$

and we complete the proof. ☐

In addition, the statistical perspectives of the findings in this research are presented. The plots of the indices are based on the formula of Theorems 3.2, 3.3, and 3.4 can be seen in Figures 2, 3, and 4. These plots use SPSS analysis.

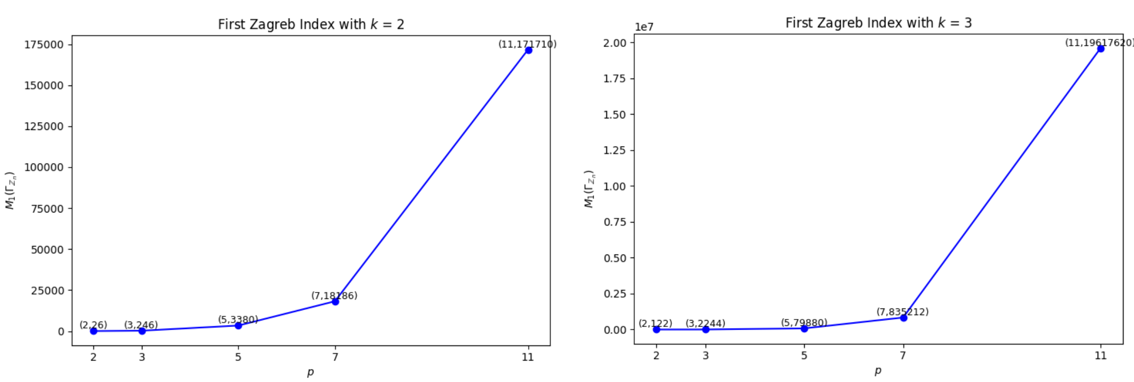

Figure 2: Graphic of First Zagreb Index, with $k = 2$ and $k = 3$.

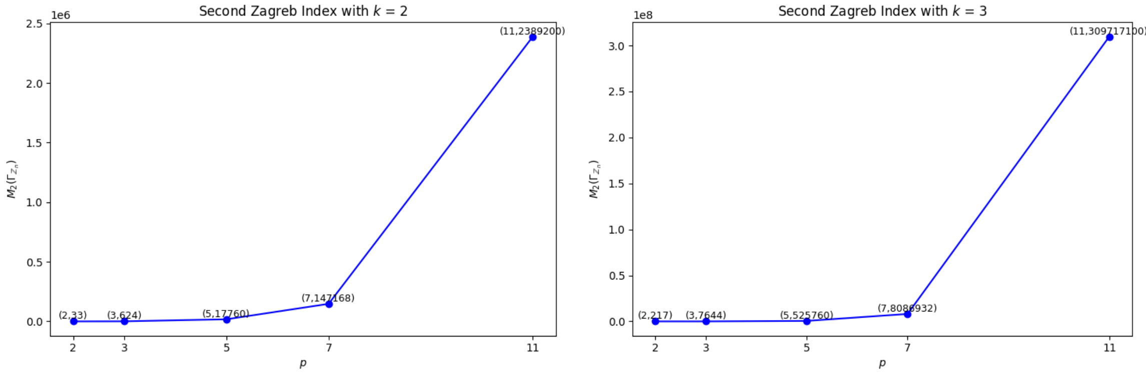

Figure 3: Graphic of Second Zagreb Index, with $k = 2$ and $k = 3$.

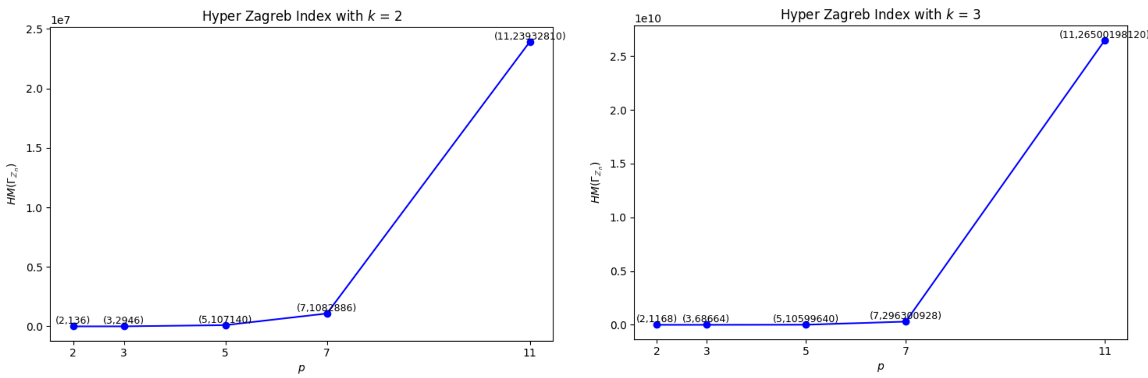

Figure 4: Graphic of Hyper-Zagreb Index, with $k = 2$ and $k = 3$.

Figures 2, 3 and 4 provide an overview of the magnitude of the Zagreb index, both the first, second, and hyper Zagreb, to make the difference in values obtained when the prime number p and integer k, for $k \geq 2$, values change more visible.

In Figure 2, the first Zagreb index for $k = 2$ and $k = 3$, the value is a greater increase starting from $p = 2$ to $p = 11$, which means the first Zagreb index will be higher for increasing p and increasing k. For $p = 11$, the first Zagreb index is higher, with a significant difference from p, before it was $(p = 7)$. Now, the second and hyper Zagreb indices, as shown in Figures 3 and 4 also have the same pattern as the first Zagreb with a higher index value as well as for p and k.

If we compare the first, second and hyper Zagreb indices for $k = 2$, and the same value p, the hyper Zagreb index is the highest when comparing the first and second Zagreb indices. In the same situation for $k = 3$, we can conclude that the hyper Zagreb index has a value much higher than the first and second Zagreb indices simultaneously for p and k. The hyper Zagreb index starts to look too high at $p = 5$, which is in contrast to the first and second Zagreb index values which start to look too high at $p = 11$.

Table 1: Results of the Zagreb Index with Spearman's Rank correlation test.

Variables	Significant	Correlation Coefficient	Information
First Zagreb Index >< Second Zagreb Index	0.000	0.996	very strong correlation
First Zagreb Index >< Hyper Zagreb Index	0.000	0.996	very strong correlation
Second Zagreb Index >< Hyper Zagreb Index	0.000	0.991	very strong correlation

The correlation test was conducted to determine the correlation between the three types of Zagreb index using the Spearman Rank correlation test. Table 1 explains that these three indexes correlate with a very strong correlation strength in a positive direction (if the value of one Zagreb index increases, then the value of the other Zagreb index also increases) with a correlation coefficient value above 0.95.This is supported by the following correlation graph between the Zagreb-based indices.

4 Conclusions

From the finding results, we have provided the first, second and hyper Zagreb indices of the prime coprime graph for group $\Gamma_{\mathbb{Z}_n}$, where $n = p^k$ and p is a prime and $k \geq 2$ is an integer, respectively. We also classified $\Gamma_{\mathbb{Z}_n}$ as a bidegreed graph; hence the connections between those indices have been presented. The indices results displayed in this paper comply with the well-known fact regarding some bounds of Zagreb-based indices. We also highlight the indices to have a strong significant correlation. The paper will constantly expose fresh questions that arise from the research, generating several opportunities for more research. Instead of Zagreb-based indices, many important properties of graphs can be explored. We might conduct a similar study for other topological indices. Since $\Gamma_{\mathbb{Z}_n}$ is a connected graph, we can extend this study by involving distance properties to construct topological indices. It is also significant to investigate the same computations for prime coprime graphs of dihedral groups.

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